Cycles in sparse graphs II

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Abstract

The *independence ratio* of a graph G is defined by

$$\iota(G) := \sup_{X \subset V(G)} \frac{|X|}{\alpha(X)},$$

where $\alpha(X)$ is the independence number of the subgraph of G induced by X. The independence ratio is a relaxation of the chromatic number $\chi(G)$ in the sense that $\chi(G) \geq \iota(G)$ for every graph G, while for many natural classes of graphs these quantities are almost equal. In this paper, we address two old conjectures of Erdős on cycles in graphs with large chromatic number and a conjecture of Erdős and Hajnal on graphs with infinite chromatic number.

1 Introduction

Let G be a graph and let $\alpha(X)$ be the size of a largest independent set in the subgraph of G induced by X. The *independence ratio* of a graph G is defined by

$$\iota(G) := \sup_{X \subset V(G)} \frac{|X|}{\alpha(X)}.$$

The independence ratio of a graph G is a relaxation of the chromatic number $\chi(G)$, since $\chi(G) \geq \iota(G)$ for all graphs G. For many interesting classes of graphs, including random and pseudorandom graphs, the chromatic number and independence ratio are equal or almost equal. On the other hand, so-called *Kneser graphs* are examples of graphs on n vertices with constant independent ratio and chromatic number of order $\log n$ [15]. In this paper, we are motivated by three conjectures on cycles in graphs with large chromatic number. We give partial evidence for the truth of each conjecture by considering graphs with large independence ratio.

1.1 Erdős' conjecture on many cycles

Erdős [5] conjectured that a triangle-free graph with chromatic number k contains cycles of at least $k^{2-o(1)}$ different lengths as $k \to \infty$. The conjecture of Erdős remains open, and in fact no lower bound better than linear in k is known for the longest cycle in a triangle-free graph with chromatic number k. In general, Theorem 2 in [20] shows that a graph of chromatic number k and no cycle of length g contains cycles of $\Omega(k^{\lceil (g-1)/2 \rceil})$ different lengths. In this paper, we prove the following theorem which shows in a very strong sense that Erdős' conjecture is true for graphs with large independence ratio:

Theorem 1 *Every triangle-free graph with independence ratio at least $k \geq 3$ has cycles of $\Omega(k^2 \log k)$ consecutive lengths.*

We prove this theorem in Section 2.5, after some preliminary results in Sections 2.1 – 2.4. The important result by Kim [13] establishing the order of magnitude of triangle-complete graph Ramsey numbers $r(3,t) = \Theta(t^2/\log t)$ shows that there are triangle-free graphs with independence ratio k and with $O(k^2 \log k)$ vertices, so the above result is best possible up to the value of the implicit constant.

1.2 Hereditary Properties

Theorem 1 is part of a more general theorem on *hereditary properties* – families of graphs closed under taking induced subgraphs. To describe the general theorem, let P be a hereditary property and let $f : [1,\infty) \to [1,\infty)$ be an increasing bijection. Then we say that P has *speed at most f* if for every $n \in \mathbb{N}$ and every n-vertex graph $G \in P$, we have $\iota(G) \leq f(n)$. Since the identity function $f(x) = x$ for $x \in [1,\infty)$ serves as an upper bound for the speed of every hereditary property, the speed of each hereditary property is well-defined. We shall prove the following theorem:

Theorem 2 *Let $f : [1,\infty) \to [1,\infty)$ be an increasing bijection. If P is a hereditary property with speed at most f, then any graph $G \in P$ with $\iota(G) > 18k + 4$ has cycles of at least $\frac{1}{2} f^{-1}(k)$ consecutive lengths.*

This theorem is proved in Section 2.6. Theorem 2 applies in general to the property P_H of H-free graphs – this is the hereditary property of graphs which do not contain any copy of H. For example, one can obtain an appropriate generalization of Theorem 1.

Theorem 3 *Let G be a K_{s+1}-free graph and suppose $\iota(G) > 18k + 4$. Then G contains cycles of at least $\frac{1}{2}(k/s)^{s/(s-1)}$ consecutive lengths.*

Clearly this theorem holds also for all H-free graphs where H has $s+1$ vertices. We prove Theorem 3 in Section 2.7. One can improve the lower bound $(k/s)^{s/(s-1)}$ in the above theorem by a polylogarithmic factor which, for $s = 3$, agrees with Theorem 1, but this involves only further computations and so will be omitted.

1.3 Erdős' conjecture on unavoidable cycles

Erdős [7] offered one thousand dollars for a satisfactory resolution of the following problem: in a graph of infinite chromatic number, which cycle lengths should appear? For example, Erdős conjectured that a graph of sufficiently large chromatic number has a cycle of length a prime. We show that if an n-vertex graph has independence ratio at least $3 \exp(8 \log^* n)$, then not only does it contain a cycle of prime length, it contains cycles of lengths from many other sparse infinite sequences of integers. Here $\log^* n$ is the number of times the natural logarithm must be applied to n to get a number less than one, and in what follows, we write $\log_b n$ for the logarithm base b, and omit the base if the logarithm is the natural logarithm. We prove the following theorem:

Theorem 4 *Let σ be an infinite increasing sequence of positive integers satisfying $\sigma_1 \geq 3$ and $\log \sigma_r \leq \sigma_{r-1}$ for all $r \geq 2$. If G is an n-vertex graph and*

$$\iota(G) \geq \sigma_1 \exp(8 \log^* n),$$

then G contains a cycle of length in the sequence σ.

Theorem 4 is proved in Section 3. We claimed that an n-vertex graph G with $\iota(G) \geq 3 \exp(8 \log^* n)$ contains a cycle of length a prime. Let p_r denote the rth prime number. Then Bertrand's Postulate gives $p_{r+1} \leq 2p_r$ for all $r \in \mathbb{N}$, and so $\log p_{r+1} \leq \log p_r + 1 \leq p_r$ for all $r \in \mathbb{N}$. Applying Theorem 4 to this sequence, we see that a graph G with $\iota(G) > 3 \exp(8 \log^* n)$ contains a cycle of length a prime, as claimed. Theorem 4 gives a similar upper bound for much sparser sequences, such as powers of three, or $2 + 1, 2^2 + 1, 2^{2^2} + 1, \ldots$ and so on.

An important remark is that Theorem 4 distinguishes between the independence ratio and the chromatic number: generalizations of Mycielski's well-known construction of triangle-free graphs of arbitrarily large chromatic number provide constructions for infinitely many n of an n-vertex graph G_n of chromatic number $\chi(G_n) = \Omega((\log n)/(\log \log n))$ with no cycle of length in a prescribed sequence σ satisfying $\log \sigma_r \leq \sigma_{r-1}$ for $r \geq 2$. The conclusion of Theorem 4 therefore does not hold if we replace $\iota(G)$ with $\chi(G)$ in the theorem. We present the details of this construction in Section 5.

1.4 Erdős-Hajnal conjecture

Let $C(G) = \{\ell : C_\ell \subset G\}$ denote the set of cycle lengths in a graph G. Erdős [7] proposed the study of the quantity

$$L(G) = \sum_{t \in C(G)} \frac{1}{t}$$

and conjectured that in a graph of infinite chromatic number, $L(G)$ diverges. This conjecture was proved by Gyárfás, Komlos and Szemerédi [12]. Specifically, they proved that if G is a finite graph of minimum degree d, then there exists $\epsilon > 0$ such that

$$L(G) \geq \epsilon \log d.$$

This is best possible up to the value of the constant ϵ, since the complete bipartite graph has $L(K_{d,d}) \leq \frac{1}{2} \log d + 1$. It follows that $L(G)$ diverges when G has infinite chromatic number, since a graph of infinite chromatic number contains a graph of minimum degree at least d for each $d \in \mathbb{N}$. The result above therefore does not rely on the chromatic number as much as the existence of subgraphs of arbitrarily large average degree. Erdős and Hajnal [7, 9] conjectured that in a graph with infinite chromatic number, the sum of reciprocals of odd cycle lengths diverges. If $C_\circ(G)$ is the set of lengths of odd cycles in G, their conjecture states that if G is a graph of infinite chromatic number, then

$$L_\circ(G) := \sum_{t \in C_\circ(G)} \frac{1}{t} = \infty.$$

In this paper, we give some evidence for this conjecture by showing:

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Theorem 5 *For any graph G on n vertices,*

$$L_\circ(G) \;\geq\; \frac{1}{2}\log \iota(G) - 8\log^* n.$$

We prove Theorem 5 in Section 4. This theorem is best possible up to the $O(\log^* n)$ term, in the sense that $L_\circ(K_t) \leq \frac{1}{2}\log \iota(K_t) + 1$. It would be interesting as a first step to the Erdős-Hajnal conjecture to show that $L_\circ(G)$ diverges when G is a graph with infinite independence ratio.

2 Preliminary results

In this section, we present the results necessary for the proofs of Theorems 1–5. The following notation will be used. If G is a graph, then $\iota(G)$ is its independence ratio and $\alpha(G)$ is its independence number. For a set $X \subset V(G)$, we denote by $\partial_G X$ the set of vertices of $V(G)\backslash X$ adjacent to at least one vertex in X. We sometimes omit the subscript G when it is clear which graph is being referred to. We write $\alpha(X)$ for the independence number of the subgraph of G induced by X. Fixing a vertex $v \in V(G)$, it is convenient to let $N_i(v)$ denote the set of vertices at distance exactly i from v.

2.1 Expanding subgraphs

The starting point for proving Theorems $1-4$ is to show that graphs with large independence ratio have nice expansion properties. Precisely, we make the following definitions:

Definition. We say that a graph G is *k-expanding on independent sets* if every independent set I in G has $|\partial_G I| > k|I|$. A graph G is *weakly k-expanding on independent sets* if for some $v \in V(G)$, every independent set $I \subset V(G)\backslash\{v\}$ has $|\partial_G I| > k|I|$.

We notice in particular that if a graph is weakly k-expanding on independent sets, then all but at most one vertex in the graph has degree more than k.

Lemma 1 *Let $k \geq 1$. Then every n-vertex graph G with $\alpha(G) < n/(k+1)$ has an induced subgraph that is k-expanding on independent sets and a 2-connected subgraph that is weakly k-expanding on independent sets.*

Proof. First we show that G has a subgraph H that is k-expanding on independent sets. Let $G_0 = G$. If G_0 has no subgraph H as above, then there is an independent set $I_0 \subset V(G_0)$ such that $|\partial I_0| \leq k|I_0|$. Let $G_1 = G - I_0 - \partial I_0$. Then there is an independent set $I_1 \subset V(G_1)$ with $|\partial I_1| \leq k|I_1|$. Let $G_2 = G_1 - I_1 - \partial I_1$. Continuing in this way, we eventually remove independent sets $I_0, I_1, \ldots, I_r$ and their neighborhoods $\partial I_0, \partial I_1, \ldots, \partial I_r$ and this exhausts all the vertices in the graph:

$$V(G_0) = \bigcup_{j=0}^{r}(I_j \cup \partial I_j).$$

However, the set $I = \bigcup_{j=0}^{r} I_j$ is an independent set of size at least $n/(k+1)$ in $G = G_0$, which is a contradiction. Therefore there exists $H \subset G$ that is k-expanding on independent sets. This proves the first statement of the lemma.

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To prove the second statement, if H is 2-connected, then we are done. If H is not 2-connected, let F be an *endblock* of H – this is a maximal 2-connected subgraph of H containing exactly one cut vertex v of H. Then for any independent set $I \subset V(F) \backslash \{v\}$,

$$|\partial_F I| = |\partial_H I| > k|I|.$$

So F is weakly k-expanding on independent sets. This completes the proof. ∎

Ajtai, Komlós and Szemerédi [1] showed that if G is an n-vertex triangle-free graph, then $\alpha(G) = \Omega(\sqrt{n \log n})$. Their result was improved by Shearer [17], who showed that if G is an n-vertex triangle-free graph of maximum degree $d \geq 2$, then

$$\alpha(G) \geq \frac{n(d \log d - d + 1)}{(d-1)^2}.$$

A straightforward calculation gives the following result:

Lemma 2 *For $n \geq e^{15}$, every n-vertex triangle-free graph G has*

$$\alpha(G) > \left(\frac{n \log n}{2}\right)^{1/2}.$$

Proof. Let $m = \left(\frac{n \log n}{2}\right)^{1/2}$. If G is a triangle-free n-vertex graph of maximum degree d, then $\alpha(G) \geq n/(d+1)$ and the lemma follows easily for $d \leq 1$. Suppose $d \geq 2$. Observing that the neighborhood of any vertex of G is an independent set, we obtain from Shearer's bound,

$$\alpha(G) \geq \max\left\{d, \frac{n(d \log d - d + 1)}{(d-1)^2}\right\}.$$

If $d > m$, then we easily recover the bound in the lemma . If $2 \leq d \leq m$, the second expression is a decreasing function of d and so it is minimized when $d = m$. In this case we get

$$
\begin{aligned}
\alpha(G) \;\; &\geq \;\; \frac{n(m \log m - m + 1)}{(m-1)^2} \\
&> \;\; \frac{n(\log m - 1)}{m} \\
&\geq \;\; \left(\frac{n \log n}{2}\right)^{1/2} + \frac{(\log \log n - 2 - \log 2)n^{1/2}}{(2 \log n)^{1/2}}.
\end{aligned}
$$

Since $n \geq e^{15}$, $\log \log n \geq 2 + \log 2$, and so we have the required bound. ∎

The second definition we require is that of expansion on arbitrary sets of vertices in a graph:

Definition. A graph G is *k-expanding on sets of size at most T* if $|\partial_G X| > k|X|$ for every $X \subset V(G)$ of size at most T. A graph G is *weakly k-expanding on sets of size at most T* if for some $v \in V(G)$, every set $X \neq \{v\}$ of size at most T has $|\partial_G X| > k|X|$.

Lemma 3 *If G is an n-vertex triangle-free graph with $\alpha(G) < n/(3k+1)$ and $k \geq e^{15}$, then G contains a 2-connected subgraph H that is weakly 2-expanding on sets of size at most $k^2 \log k$ and weakly $3k$-expanding on independent sets.*

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Proof. Pass to a 2-connected subgraph $H \subset G$ which is weakly $3k$-expanding on independent sets, using Lemma 1. Then for some $v \in V(H)$, every independent set $I \subset V(H)\backslash\{v\}$ has $|\partial_H I| > 3k|I|$. Let $X \subset V(H)$ satisfy $|\partial_H X| \leq 2|X|$ where $X \neq \{v\}$. If $u \in X\backslash\{v\}$, then

$$|\partial_H X| \geq |\partial_H\{u\}| - |X| > 3k - |X|.$$

If $|X| < e^{15}$, then since $k \geq e^{15}$ we obtain $|\partial_H X| > 2|X|$. If $|X| \geq e^{15}$, then $H[X]$ is a triangle-free graph with at least e^{15} vertices. Suppose $|X\backslash\{v\}| = x$. By Lemma 2, there is an independent set $I \subset X\backslash\{v\}$ such that

$$|I| \geq \left(\frac{x \log x}{2}\right)^{1/2} \geq 3,$$

Consequently, $|X| - |I| \leq x - 2$ and therefore

$$
\begin{aligned}
2|X| = 2x + 2 &\geq |\partial_H X| \\
&\geq |\partial_H I| - |X| + |I| \\
&> 3k|I| - x + 2 \\
&\geq 3k \cdot \left(\frac{x \log x}{2}\right)^{1/2} - x + 2.
\end{aligned}
$$

This implies $2x > k^2 \log x$ and therefore $x > k^2 \log k$. This completes the proof. ∎

2.2 Pósa's lemma and long cycles

A well-known result of Pósa [16] shows how to find long paths in graphs which are 2-expanding on sets of size at most T: in this case one obtains a path of length at least $3T$. To describe Pósa's Lemma and the variant we use, we require some notation. If $P = v_1 v_2 \ldots v_m$ is a longest path in a graph G and $\{v_1, v_i\}$ is an edge of G, consider a path $Q = v_{i-1} \ldots v_1 v_i \ldots v_m$ of the same length as P, obtained by adding edge $\{v_1, v_i\}$ and deleting edge $\{v_{i-1}, v_i\}$ from P. We say that Q was obtained from P via an *elementary rotation*, which keeps endpoint v_m fixed. The set of all vertices of P which are endpoints of paths obtained by repeated elementary rotations from P with fixed endpoint v_m is denoted by $S(P)$. The following variant of Pósa's Lemma (see Lemma 2.7 in [3]) is our starting point:

Proposition 1 *Let $T \geq 1$, and let G be a graph that is 2-expanding on sets of size at most T. Then for any longest path $P \subset G$ there is a cycle $C \subset H$ of length at least $3T$ containing $S(P) \cup \partial S(P)$.*

We require a slight adjustment of this proposition to accommodate weak 2-expansion: recall that G is weakly 2-expanding on sets of size at most T if for some $v \in V(G)$ and every set $X \subset V(G)$ with $X \neq \{v\}$, $|\partial_G X| > 2|X|$. The proof of the proposition below is almost identical to the proof of Proposition 1 given in [3].

Proposition 2 *Let $T \geq 1$, and let G be a graph that is weakly 2-expanding on sets of size at most T. Then there exists a longest path $P \subset G$ and a cycle $C \subset H$ of length at least $3T$ containing $S(P) \cup \partial S(P)$.*

Proof. By definition of weak expansion, for some $v \in V(G)$ and every $X \subset V(G)$ with $X \neq \{v\}$ we have $|\partial_G X| > 2|X|$. Let P be a longest path in G and let $v_1 \neq v$ be an endpoint of P and v_m the

other endpoint. The crucial step in the proof of Proposition 1 is that $\partial S(P) \subset S(P)^- \cup S(P)^+$, where $S(P)^-$ is the set of vertices preceding $S(P)$ on P and $S(P)^+$ is the set of vertices succeeding $S(P)$ on P. Since $|S(P)^-| \leq |S(P)|$ and $|S(P)^+| \leq |S(P)|$, we must have $|\partial S(P)| \leq 2|S(P)|$. Since G is weakly 2-expanding on sets of size at most T, we must have $S(P) = \{v\}$ or $|S(P)| > T$. However since $v_1 \neq v$, $|\partial_G\{v_1\}| > 2$ which shows that $|S(P)| > 1$ and therefore $S(P) \neq \{v\}$. It follows that $|S(P)| > T$. To construct a cycle containing $S(P) \cup \partial S(P)$, let y be the last vertex of $\partial S(P)$ on P. The segment of P from v_1 to y contains all vertices of $S(P) \cup \partial S(P)$, otherwise any vertex of $S(P)$ after y on P would be distinct from v_m and then the vertex after it on P would be an element of $\partial S(P)$, contradicting that y is the last vertex of $\partial S(P)$ on P. If $x \in S(P)$ is a neighbor of y in G, and Q is a path from x to v_m obtained by elementary rotations from P, then Q contains the segment of P from y to v_m. Moreover, starting from x, Q traverses all vertices of $S(P) \cup \partial S(P)$ before reaching y and then continues along segment of P from y to v_m. So Q together with the edge $\{x, y\}$ forms the required cycle. Since $|S(P) \cup \partial S(P)| \geq 3T$, this cycle has the required length. ∎

Combining Lemma 3 with Proposition 2, we arrive at the following theorem, which shows that triangle-free graph with large independence ratio contains a very long cycle.

Theorem 6 *Let G be a triangle-free graph with $\iota(G) > 3k + 1$ where $k \geq e^{15}$. Then G has a cycle of length at least $3k^2 \log k$.*

Proof. By Lemma 3, there is a 2-connected graph $H \subset G$ that is weakly 2-expanding on sets of size at most $k^2 \log k$. By Proposition 2, there is a cycle $C \subset H$ of length at least $3k^2 \log k$ in G, completing the proof. ∎
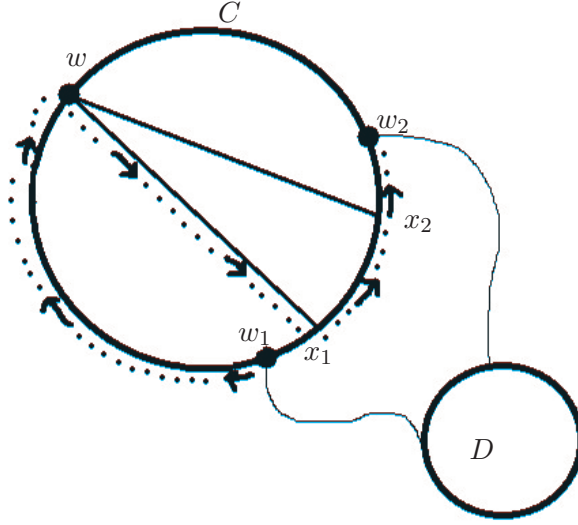

2.3   Long odd cycles with chords

A *chord of a cycle* is an edge which joins two non-consecutive vertices of the cycle. To prove Theorem 1, we need to extend Theorem 6 further to obtain a long odd cycle with a chord.

Proposition 3 *Let $k \geq e^{15}$ and let G be an n-vertex triangle-free graph with $\alpha(G) < n/(3k + 1)$. Then G contains a non-bipartite graph consisting of a cycle of length at least $\frac{1}{2}k^2 \log k$ with at least one chord.*

Proof. By Lemma 3, G contains a 2-connected subgraph H that is weakly 2-expanding on sets of size at most $k^2 \log k$ and weakly $3k$-expanding on independent sets. By Proposition 2, there is a cycle $C \subset H$ of length at least $3T := 3k^2 \log k$ containing $S(P) \cup \partial_H S(P)$ for some longest path P with $|S(P)| \geq T$. The vertices of $S(P)$ are called *special vertices*, and since $\partial_H S(P) \subset V(C)$, all their neighbors are vertices of C. Since H is weakly $3k$-expanding on independent sets, all but at most one vertex of H has degree more than $3k \geq e^{15}$. In particular, every special vertex has two neighbors on C which are not adjacent to that special vertex on C. If $V(C)$ induces a non-bipartite subgraph of H, then we are done – take C together with an appropriate chord of C. Suppose $V(C)$ induces a bipartite subgraph of H. Since H is weakly $3k$-expanding on independent sets, $\chi(H) \geq \iota(H) > 3k \geq e^{15}$. In particular, $H - V(C)$ is certainly non-bipartite and contains an odd cycle D. By 2-connectivity of H, there exist two vertex disjoint paths Q'_1 and Q'_2 from $V(D)$ to $V(C)$ whose internal vertices are in $H - V(C) - V(D)$. In particular, there exist $w_1, w_2 \in V(C)$ such that there are both even and odd length

paths Q_1 and Q_2 respectively and $V(Q_1) \cap V(C) = \{w_1, w_2\} = V(Q_2) \cap V(C)$. Indeed, let Q_i consist of $Q_1' \cup Q_2'$ together with a subpath of D of length congruent to $|E(Q_1')| + |E(Q_2')| + i$ modulo two, for $i \in \{1, 2\}$. Let P_1 and P_2 denote the two $w_1 w_2$-subpaths of C, and suppose P_1 contains at least $\frac{1}{2}T$ special vertices. If P_1 has a chord, then $P_1 \cup Q_1$ or $P_1 \cup Q_2$ is the required non-bipartite subgraph. If P_1 has no chord, then at least $\frac{1}{2}T$ special vertices on P_1 each have at least two neighbors in $V(P_2) \backslash \{w_1, w_2\}$. Pick a special vertex $w \in V(P_1)$ and neighbors $x_1, x_2 \in N(w) \cap V(P_2)$. We assume that the order of appearance of vertices on C is w_1, w, w_2, x_2, x_1 clockwise. For $a, b \in V(C)$, let $C(a, b)$ denote the internal vertices of the subpath of C from a to b in the clockwise order where a precedes b. Consider the paths $R_1 = C - C(x_1, w_1) - C(w, w_2) + \{w, x_1\}$ and $R_2 = C - C(w_2, x_2) - C(w_1, w) + \{w, x_2\}$. The path R_1 is shown by dotted lines and arrows in the figure below. Then $R_1 \cup R_2 = C$ and $\{w, x_2\}$ is a chord of R_1 and $\{w, x_1\}$ is a chord of R_2. One of these paths has length at least $\frac{1}{2}|V(C)| > T$, and together with Q_1 or Q_2 forms the required non-bipartite subgraph. ∎

2.4 Consecutive cycle lengths

The main ingredient for finding cycles of many consecutive lengths in graphs is the following proposition [21]:

Proposition 4 *Let F be a non-bipartite graph comprising a cycle with a chord, and let (A, B) be a non-trivial partition of $V(F)$. Then for each $\ell \leq |V(F)| - 1$, there exists a path in F of length ℓ with one endpoint in A and the other endpoint in B.*

We illustrate the argument which will be used repeatedly in the rest of the paper. Recall that $N_i(v)$ denotes the set of vertices at distance exactly i from a given vertex v in a graph.

Proposition 5 *Let G be a graph and suppose that $N_i(v)$ contains a non-bipartite graph F comprising a cycle of length L together with a chord. Then G contains cycles of L consecutive lengths, the shortest of which has length at most $2i + 1$.*

Proof. Consider a breadth-first search tree rooted at v, and let T be a minimal subtree whose set of leaves is $V(F)$. Then T branches at its root, and we let A be the set of leaves of T in one branch, and B the remaining set of leaves of T. Then $A \cup B = V(F)$ and A, B partition $V(F)$. Now we use Proposition 4: there exists a path in F of length ℓ whose first vertex is in A and whose last vertex is in B for all $\ell \leq |V(F)| - 1$. If T has height h, then we obtain cycles of lengths $2h+1, 2h+2, \ldots, 2h+L-1$ in H, as required. Furthermore, since $h \leq i$, the shortest cycle has length $2h + 1 \leq 2i + 1$. ∎

2.5   Proof of Theorem 1

Theorem 1 states that a triangle-free graph G with $\iota(G) \geq k \geq 3$ contains cycles of $\Omega(k^2 \log k)$ consecutive lengths. It is enough to prove the following quantitative version of this statement: let G be a triangle-free graph with $\iota(G) > 18k + 4$ where $k \geq e^{15}$, then G contains cycles of at least $k^2 \log k$ consecutive lengths. Pass to a subgraph F of G such that $\alpha(F) < |V(F)|/(18k + 4)$. By Lemma 1, F has a subgraph H that is $18k + 3$-expanding on independent sets. In this subgraph, pick any vertex v and consider $N_i := N_i(v)$ for $i \in \mathbb{N}$. If for some $i \in \mathbb{N}$, $\iota(N_i) > 3(2k) + 1$, then Proposition 3 shows that the subgraph induced by $N_i(v)$ contains a non-bipartite graph consisting of an odd cycle of length at least $\frac{1}{2}(2k)^2 \log 2k > k^2 \log k$ with at least one chord. In this case, the theorem follows from Proposition 5. Otherwise, we have $\iota(N_i) \leq 6k + 1$ for every $i \in \mathbb{N}$. In that case, $\alpha(N_i) \geq |N_i|/(6k + 1)$ and N_i contains and independent set I of size at least $|N_i|/(6k + 1)$. Since H is $18k + 3$-expanding on independent sets, we conclude $|\partial I| > 3|N_i|$. In particular, since

$$\partial I \subset N_i \cup N_{i-1} \cup N_{i+1},$$

for all $i \in \mathbb{N}$ we have

$$|N_{i+1}| + |N_{i-1}| > 3|N_i| - |N_i| = 2|N_i|.$$

Then $|N_0| = 1$ and, since H is $18k + 3$-expanding on independent sets, every vertex of H has degree more than $18k + 3$, so $|N_1| > 18k + 3 > |N_0|$. We easily obtain $|N_{i+1}| > |N_i|$ by induction for all $i \in \mathbb{N}$, which is clearly impossible since H is a finite graph so some N_i must be empty. This completes the proof. ∎

2.6   Hereditary Properties

In this subsection we prove Theorem 2. The proof of Theorem 2 is almost identical to the proof of Theorem 1 given over the last three sections, so we merely indicate how to generalize each component of that proof. Throughout this section, $k \geq 1$ and $f : [1, \infty) \to [1, \infty)$ is an increasing bijection and P is a hereditary property. For a general hereditary property, the following lemma generalizes Lemma 3.

Lemma 4 *Let P denote any hereditary property with speed at most f and $G \in P$ where $|V(G)| = n$ and let $k \geq 1$. If $\alpha(G) < n/(6k + 1)$ then G has a 2-connected subgraph that is weakly 2-expanding on sets of size at most $f^{-1}(k)$ and weakly $3k$-expanding on independent sets.*

Proof. We repeat the proof of Lemma 3. Pass to a 2-connected subgraph $H \subset G$ which is weakly $6k$-expanding on independent sets, using Lemma 1: for some $v \in V(H)$ and every independent set $I \subset V(H) \backslash \{v\}$ we have $|\partial_H I| > 6k|I|$. Let $X \subset V(H)$ with $X \neq \{v\}$, and suppose $|\partial_H X| \leq 2|X|$. If $X = \{x\}$, then $x \neq v$ and $|\partial_H \{x\}| > 6k > 2|X|$ so we have $|X| \geq 2$. Since P is a hereditary property

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of speed at most f, there is an independent set $I \subset X \backslash \{v\}$ such that $|I| \geq (|X| - 1)/f(|X| - 1)$, since the subgraph of H induced by X is in P. Consequently,

$$2|X| \geq |\partial_H X| \geq |\partial_H I| - |X| + |I| > 6k|I| - |X|$$

and so $2k|I| < |X|$. So if $|\partial_H X| \leq 2|X|$, then

$$2k(|X| - 1)/f(|X| - 1) \leq 2k|I| < |X|$$

from which we get $f(|X| - 1) > 2k(|X| - 1)/|X| \geq k$ since $|X| \geq 2$. It follows that $|X| > f^{-1}(k)$. ∎

Using this lemma, we obtain the following straightforward generalization of Proposition 3 for hereditary properties.

Proposition 6 *Let P be a hereditary property with speed at most f and $G \in P_n$ and let $k \geq 1$. If $\alpha(G) < n/(6k + 1)$, then G contains a non-bipartite graph consisting of a cycle of length at least $\frac{1}{2}f^{-1}(k)$ with at least one chord.*

Proof of Theorem 2. Let $G \in P$ have $\iota(G) > 18k + 4$ where $k \geq 1$. By Lemma 1, G has a subgraph H that is $18k + 3$-expanding on independent sets. In this subgraph, pick any vertex v and consider $N_i = N_i(v)$ for $i \in \mathbb{N}$. If for some $i \in \mathbb{N}$, $\iota(N_i) > 6k + 1$, then Proposition 6 shows that the subgraph induced by $N_i(v)$ contains a non-bipartite graph F consisting of a cycle of length at least $\frac{1}{2}f^{-1}(k)$ with at least one chord. In this case, the theorem follows from Proposition 5. Otherwise, we have $\iota(N_i) \leq 6k + 1$ for every $i \in \mathbb{N}$. In that case, $\alpha(N_i) \geq |N_i|/(6k + 1)$. Since H is $18k + 3$-expanding on independent sets, we conclude $|N_{i-1} \cup N_i \cup N_{i+1}| > 3|N_i|$, which leads to the contradiction $|N_i| > |N_{i-1}|$ for all $i \in \mathbb{N}$, as in Theorem 1. ∎

2.7 The property of H-free graphs

To prove Theorem 3, we compute an upper bound for the speed of P_{s+1}, the family of K_{s+1}-free graphs. Note that if a graph contains K_{s+1}, then it contains a copy of every graph H on $s + 1$ vertices, and in this case $P_H \subset P_{s+1}$. We bound the speed using a well-known upper bound on the Ramsey number $r(K_{s+1}, K_{t+1})$:

Proposition 7 *If a graph G has at least $\binom{s+t}{s}$ vertices, then it contains a clique of order $s + 1$ or an independent set of order $t + 1$. In particular, P_{s+1} has speed at most f_s where $f_s(x) = \min\{x, sx^{1-1/s}\}$ for $x \in [1, \infty)$.*

Proof. The first statement is a well-known bound on Ramsey numbers $r(K_{s+1}, K_{t+1})$. Note that $\binom{s+t}{s} \leq (st)^s$ for all $s, t \geq 1$. It follows that for any m-vertex K_{s+1}-free graph H, there is an independent set of size at least $t + 1$ in H whenever $m \geq (st)^s$. By definition, an n-vertex graph K_{s+1}-free graph G has

$$\iota(G) = \sup_{X \subset V(G)} \frac{|X|}{\alpha(X)} \leq \sup_{1 \leq (st)^s \leq n} \frac{(st)^s}{t + 1} < sn^{1-1/s}.$$

We also have $\iota(G) \leq n$. This completes the proof. ∎

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For each fixed s, the function f_s in the last proposition is a continuous increasing function on $[1, \infty)$ with $f_s(1) = 1$, and therefore $f_s : [1, \infty) \to [1, \infty)$ is an increasing bijection. Applying Theorem 2, if $G \in P_{s+1}$ and $\iota(G) > 18k + 4$, then G contains cycles of at least $\frac{1}{2}f^{-1}(k)$ consecutive lengths. Since

$$f^{-1}(k) = \max\{k, (k/s)^{s/(s-1)}\} \geq (k/s)^{s/(s-1)}$$

we conclude that a graph $G \in P_{s+1}$ with $\iota(G) > 18k+4$ has cycles of at least $\frac{1}{2}(k/s)^{s/(s-1)}$ consecutive lengths. This completes the proof of Theorem 3. $\square$

Remark. By using better bounds on $r(K_{s+1}, K_{t+1})$, Theorem 3 can be improved by logarithmic factors of k, as we achieved for triangles. However computing bounds on the speed of P_{s+1} is then more cumbersome, so for simplicity we avoid these calculations.

3   Proof of Theorem 4

We are given a sequence σ_r satisfying $\log \sigma_r \leq \sigma_{r-1}$, and we have to prove that any graph G with $\iota(G) > \sigma_1 \exp(8 \log^* n)$ contains a cycle of length σ_r for some r. We begin with some notation. Let τ denote any infinite increasing subsequence of σ with $\tau_1 = \sigma_1$ and let P_r denote the property of all graphs containing no cycle of any length $\sigma_j \leq \tau_r$. Define

$$\triangle_r := \max\{\sigma_j - \sigma_{j-1} : \sigma_j \leq \tau_r\}.$$

We define $\tau_0 = 1 = \triangle_1$. To prove Theorem 4, we first prove the following more general theorem:

Theorem 7 *Let σ be an infinite increasing sequence of positive integers with $\sigma_1 \geq 3$ and let τ be an arbitrary subsequence of σ with $\tau_1 = \sigma_1$ and $\triangle_r$ defined as above. Then any n-vertex graph $G \in P_r$ has*

$$\iota(G) < 27^r \sigma_1 \exp\left(\sum_{i=1}^{r} \frac{2 \log \triangle_i}{\tau_{i-1}} + \frac{2 \log n}{\tau_r}\right).$$

Before proving this theorem, we show how to derive Theorem 4 from it:

Proof of Theorem 4. In Theorem 4, we are given a sequence σ with $\log \sigma_r \leq \sigma_{r-1}$ for $r \geq 2$. Let $T(r)$ denote the rth element of the sequence

$$\sigma_1 \quad e^{\sigma_1} \quad e^{e^{\sigma_1}} \quad \cdots$$

Since $\log \sigma_r \leq \sigma_{r-1}$, there is an element σ between $T(r-1)$ and $T(r)$ for all $r \geq 2$, which we define to be τ_r. We now apply Theorem 4 with this sequence $\tau = (\tau_r)_{r \in \mathbb{N}}$ and $\tau_1 = \sigma_1$. Then choose the smallest value of r such that $\tau_r > 2 \log n$. Since $T(r)$ is at least a tower of es of height r, we note that $r \leq \log^* n$. The key fact is that

$$\sum_{i=1}^{r} \frac{2 \log \triangle_i}{\tau_{i-1}} \leq 2r$$

since $\log \triangle_r \leq \tau_{r-1}$ for all $r \geq 2$. In conclusion, from Theorem 7, we have

$$\iota(G) < 27^r \sigma_1 \exp(2r + 1) < \sigma_1 \exp(8 \log^* n)$$

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and this completes the proof. □

Proof of Theorem 7. Theorem 7 is a consequence of the following claim:

Claim 1. *Let $r, m \in \mathbb{N}$ and $\delta_r = 1/\lceil \tau_r/2 \rceil$. Then every m-vertex graph $G \in P_r$ has*

$$\alpha(G) \geq \frac{1}{a_r} m^{1-\delta_r}.$$

where a_r is defined by

$$a_1 = 27\sigma_1 \quad and \quad a_r = 27 a_{r-1} \triangle_r^{\delta_{r-1}}.$$

To see how Theorem 7 follows, take a graph $G \in P_r$ with n vertices; then Claim 1 shows

$$\iota(G) \leq \sup_{X \subset V(G)} \frac{|X|}{\alpha(X)} \leq \sup_{X \subset V(G)} \frac{|X|}{\frac{1}{a_r}|X|^{1-\delta_r}} = a_r n^{\delta_r}$$

since we can apply Claim 1 with $m = |X|$ to the subgraph of G induced by each set $X \subset V(G)$. The linear recurrence $a_r = 27 a_{r-1} \triangle_r^{\delta_{r-1}}$ gives

$$a_r = 27^r \sigma_1 \prod_{i=1}^{r} \triangle_i^{\delta_{i-1}} \leq (27)^r \sigma_1 \exp\left(\sum_{i=1}^{r} \frac{2\log \triangle_i}{\tau_{i-1}}\right)$$

and therefore

$$\iota(G) \leq a_r n^{\delta_r} \leq 27^r \sigma_1 \exp\left(\sum_{i=1}^{r} \frac{2\log \triangle_r}{\tau_{r-1}} + \frac{2\log n}{\tau_r}\right).$$

We proceed to the proof of Claim 1.

Proof of Claim 1. The claim is true for P_1 using early known bounds on cycle-complete Ramsey numbers [8] (see also [21]): for $\sigma \in \{2\ell + 1, 2\ell + 2\}$,

$$r(C_L, K_t) < 27\sigma_1 t^{1+1/\ell}$$

from which we obtain the required lower bound. Suppose $r > 1$ and that Claim 1 has been proved for every graph in P_{r-1}, but that the claim is false for P_r. We first compute an upper bound for the speed of P_{r-1}.

Claim 1.1. *The property P_{r-1} has speed at most f where*

$$f(x) = \min\{x, a_{r-1} x^{\delta_{r-1}}\} \quad for \ x \in [1, \infty).$$

Proof of Claim 1.1. Since Claim 1 holds for P_{r-1}, we have that every p-vertex graph $F \in P_{r-1}$ satisfies

$$\alpha(F) \geq \frac{1}{a_{r-1}} p^{1-\delta_{r-1}}$$

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This implies
$$\iota(F) = \sup_{X \subset V(G)} \frac{|X|}{\alpha(X)} \leq a_{r-1}p^{\delta_{r-1}}.$$

Therefore P_{r-1} has speed at most f where
$$f(x) = \min\{x, a_{r-1}x^{\delta_{r-1}}\}$$

for $x \geq 1$, as required. $\qquad\square$

Since by assumption Claim 1 does not hold for P_r, for some $m \in \mathbb{N}$ there exists an m-vertex graph $G \in P_r$ such that
$$\alpha(G) < \frac{1}{a_r}m^{1-\delta_r}.$$

Using Lemma 1, pass to an induced subgraph H of G which is $a_r m^{\delta_r}$-expanding on independent sets. For the rest of the proof of Claim 1, we work in the subgraph H of G to derive a contradiction.

Claim 1.2. *For any $v \in V(H)$ and any positive integer $j \leq 1/\delta_r$,*
$$\alpha(N_j(v)) > \frac{3|N_j(v)|}{a_r}.$$

Proof of Claim 1.2. Fix $j \leq 1/\delta_r$ and let H_j be the subgraph of H induced by $N_j(v)$. For convenience, let $h_j = |N_j(v)| = |V(H_j)|$. If $\alpha(H_j) > h_j/9$, then the proof of Claim 1.2 is complete, since $a_r \geq 9\sigma_1 \geq 27$ for all $r \in \mathbb{N}$. We may therefore write $\alpha(H_j) = h_j/(6k+1)$ where $k \geq 1$ is a real number. By Proposition 6, if f is an upper bound for the speed of P_{r-1}, then H_j contains a non-bipartite graph comprising a cycle of length at least $\frac{1}{2}f^{-1}(k)$ with a chord. It is important to note that $P_r \subset P_{r-1}$ and therefore $H_j \in P_{r-1}$. By Claim 1.1,
$$f^{-1}(k) = \max\{k, (k/a_{r-1})^{1/\delta_{r-1}}\} \geq (k/a_{r-1})^{1/\delta_{r-1}}.$$

By Proposition 5, H contains cycles of at least $\frac{1}{2}f^{-1}(k)$ consecutive lengths, the shortest of which has length at most $2j + 1 \leq \tau_r$. By definition of $\triangle_r$, and since $G \in P_r$, we must have $\frac{1}{2}f^{-1}(k) \leq \triangle_r$. Recall that $r \geq 2$ and therefore $\delta_{r-1} \leq 1/2, a_{r-1} \geq 81$. Rearranging this inequality, and using $\alpha(H_j) = h_j/(6k+1)$, we obtain
$$\alpha(H_j) > \frac{h_j}{9a_{r-1}\triangle_r^{\delta_{r-1}}}.$$

By definition of a_r, the denominator is $a_r/3$, as required. $\qquad\square$

Since H is $a_r m^{\delta_r}$-expanding on independent sets, we have for any $j \leq 1/\delta_r$ and any maximum independent set I in H_j,
$$|\partial_H N_j(v)| \geq |\partial_H I| > 3m^{\delta_r} \cdot h_j.$$

Since $\partial_H N_j(v) \subset N_j(v) \cup N_{j-1}(v) \cup N_{j+1}(v)$, we conclude that
$$h_{j+1} + h_{j-1} > (3m^{\delta_r} - 1)h_j \geq 2m^{\delta_r}h_j.$$

We also have $h_0 = 1$ and $h_1 > 2m^{\delta_r}$, since H is $a_r m^{\delta_r}$-expanding on independent sets. Now the recurrence inequality $h_j + h_{j-1} > (2c)h_j$ with $h_0 = 1$ and $h_1 > 2c$ has $h_j > c^j$ for all j. With $c = m^{\delta_r}$, we obtain $h_j > m^{j\delta_r}$ for all $j \leq 1/\delta_r$, and in particular we obtain the contradiction $h_j > m = |V(G)|$ when $j = 1/\delta_r$. This completes the proof of Claim 1, and hence Theorem 7. $\qquad\square$

4   Proof of Theorem 5

We are given an n-vertex graph G with $\iota(G) \geq t$ and we have to show

$$L_\circ(G) \geq \frac{1}{2} \log t - 8 \log^* n.$$

This is clearly true if $t \leq \exp(16 \log^* n)$ so we assume $t > \exp(16 \log^* n)$. We let $s = t/\exp(8 \log^* n)$, and consider the disjoint intervals of odd numbers:

$$S_i = [s^i, s^{i+1}) \cap (2\mathbb{N} + 1)$$

for $i \geq 0$. If for some i we have $S_i \subset C(G)$, then

$$L_\circ(G) \geq \frac{1}{2} \log s \geq \frac{1}{2} \log t - 8 \log^* n$$

and the proof is complete. Otherwise, for each i we pick $\sigma_i \in S_i \backslash C(G)$. Then we have defined a sequence σ with $\sigma_i \leq s\sigma_{i+1}$. Let τ be a subsequence of σ such that $\log \tau_r \leq \tau_{r-1}$ and such that τ_r is contained in the interval S_i for which $\sigma_1 T(r) \geq s^{i+1} - 1$ but $\sigma_1 T(r) < s^{i+2} - 1$. Here $T(r)$ is a tower of es of height r. Note that τ_r is well defined, since intervals S_i cover all numbers. We also let $\tau_1 = \sigma_1$. Applying Theorem 7 with this sequence τ and choosing r such that $\tau_r > 2 \log n$, we have $r \leq \log^* n$ and

$$\iota(G) < 27^r \sigma_1 \exp(2r + 1) < \sigma_1 \exp(8 \log^* n) < s \exp(8 \log^* n) < t.$$

This contradiction completes the proof.   □

5   Constructions

In this section, we give constructions which show that the conclusion of Theorem 4 does not hold if $\iota(G)$ is replaced with the chromatic number $\chi(G)$.

Construction. The existence of triangle-free graphs with arbitrarily large chromatic number was explicitly established by the so-called Mycielski graphs. A survey is given in [19]. These constructions were generalized (see page 213 in [19]) to give, for each $k, r \in \mathbb{N}$ where r is odd, a graph $G_{k,r}$ with chromatic number k and no odd cycle of length at most r and with $|V(G_{k,r})| = 2^{3-k}(r+2)^{k-2}$. We observe that $|V(G_{k,r})| \leq r^k - 1$ for every $r, k \geq 3$, so we let $G'_{k,r}$ consist of $G_{k,r}$ together with enough isolated vertices so that $|V(G'_{k,r})| = r^k - 1 := n$. Note that n is even. Provided that $\log(n+1) \leq r$, the increasing sequence σ of odd integers in $\{3, 5, \ldots, r\} \cup \{n+1, n+3, \ldots\}$ satisfies the requirements of Theorem 4 and yet $G_{k,r}$ has no cycle of length in σ. For instance, if we take $k = \lfloor r/\log r \rfloor$ and $k \geq 3$, then $n + 1 = r^k \leq e^r$ so $\log(n+1) \leq r$, and

$$\chi(G_{k,r}) = k \geq \left\lfloor \frac{\log(n+1)}{\log \log(n+1)} \right\rfloor \gg \exp(8 \log^* n).$$

It follows that while $\chi(G_{k,r})$ is substantially larger than the bound $\exp(8 \log^* n)$ for the independence ratio in Theorem 4, $G_{k,r}$ has no cycles of length in the sequence σ.

6 Concluding remarks

• We remark that the chromatic number can be arbitrarily large relative to the independence ratio of a graph. Consider the *Kneser graph* $K_{n:r}$ whose vertex set is all subsets of $\{1, 2, \ldots, n\}$ of size r, and whose edges consisting of pairs of disjoint subsets of $\{1, 2, \ldots, n\}$ of size r. By Lovász's Theorem [15], $\chi(K_{n:r}) = n - 2r + 2$. By the Erdős-Ko-Rado Theorem [10], $\alpha(K_{n:r}) = \binom{n-1}{r-1}$ and therefore $\iota(K_{n:r}) \geq n/r$. If $n = sr$ where $s > 2$ then for any set $X \subset K_{n:r}$, we claim $\alpha(X) \leq |X|/s$. To see this, given any collection of $|X|$ subsets of $\{1, 2, \ldots, sr\}$ of size r, there exists an $i \in \{1, 2, \ldots, sr\}$ such that i is contained in at least $r|X|/sr = |X|/s$ of the sets of size r. The sets containing i therefore form an independent set of size at least $|X|/s$ in $K_{n:r}$. Consequently $\iota(K_{sr:r}) \leq s$ and we conclude $\iota(K_{sr:r}) = s$. Writing $|V(K_{sr:r})| = N$, we obtain $N = \binom{sr}{r} < (es)^r$ and therefore

$$\chi(K_{n:r}) > \frac{\iota(G) - 2}{\log \iota(G) + 1} \log N.$$

A good example is the Kneser graph $K_{3r,r}$ with $N = \binom{3r}{r}$ vertices, which has independence ratio three and chromatic number $\Theta(\log N)$.

• The Erdős-Hajnal conjecture [7, 9] remains open. A partial step would be to show that if G is a graph with infinite independence ratio, then $L_\circ(G)$ is infinite, whereas in this paper we showed $L_\circ(G) > \frac{1}{2} \log \iota(G) - 8 \log^* n$ when G is an n-vertex graph. Perhaps it is true that $L_\circ(G) > \frac{1}{2} \log \chi(G) - O(1)$ for any finite graph G, although this is an even stronger question than the Erdős-Hajnal [9] conjecture.

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